Infinite dimensional dynamics associated to quadratic Hamiltonians

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Abstract

We study here $\mathbb{R}^{\mathbb{Z}^d}$-valued gradient diffusions associated to quadratic interactions. We establish that each each Gaussian Gibbs measure associated to this interaction can be obtained as limit in time of the solution of the linear diffusion for a set of initial deterministic conditions which we describe. Thus the absence of phase transition corresponds to the ergodicity of the system. Moreover, we study the influence of a phase transition on the speed of convergence. Finally, we prove that the invariant measures for these gradient diffusions are exactly the associated Gibbs measures.

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Introduction

The study of lattice random fields with given conditional Gaussian distributions was initiated in 1966 by Dobrushin, who built mathematical fundations for the theory of Gibbs measures. In 1980, he proved that these measures are obtained as mixture of Gaussian fields with suitable covariance and mean value. At the same time, the work of Glauber for discrete spin systems stimulated the study of connections between infinite gradient type stochastic differential equations and lattice Gibbs measures (see Royer [23]). The challenge of statistical mechanicians is to identify the three following sets : Gibbs measures, reversible measures, and stationary measures for the associated gradient dynamics. It is also interesting to get some informations about the asymptotic behaviour of the dynamics.

Nowadays, one can assure the identification between reversible measures of infinite-dimensional diffusions and Gibbs measures : the pioneer work has been performed by Doss and Royer [8], and, more recently, Cattiaux, Rœlly and Zessin [4] have used a new approch based on the study of Gibbs states on the trajectory space $\mathcal{C}[0,T]^{\mathbb{Z}^d}$. But the conjecture that the stationary measures under gradient dynamics are Gibbs is unfortunately yet far from completeness : Holley and Strook [15] studied a special class of symmetric diffusion processes on the denumerable product of torus $\mathbb{U}^{\mathbb{Z}^d}$ and Fritz [10] obtained results for translation-invariant measures and some superstable interactions on the one or two dimensional lattice, but without intersection with the Gaussian case.

In spite of the fact that the gradient dynamics associated to quadratic interactions are linear ones, the actual knowledge leaves unsolved the asymptotic behaviour for nonzero initial conditions in case of phase transition : it is then one of our new results, presented in this paper – we will see that in the Gaussian case, phase transition and absence of spectral gap occurs simultaneously.

The first section presents the technical background of our work : we introduce the diffusion equation, the state space in which the process lives and assumptions satisfied by the potential. We show existence and uniqueness of the linear diffusion equation in the appropriate space and compute the covariance. In section 2, we study the time asymptotic behaviour of the solution. We prove that every extremal Gibbs measure can be obtained as a limit of the system in infinite time for a deterministic initial condition : we exhibit a subset of the domain of attraction and compute the speed of convergence. In the last section, we prove that every stationary measure for the gradient linear dynamics is a Gibbs measure.

1 Framework

1.1 The diffusion equation and its state space

We denote by W_t a family $(W_t^i)_{i\in\mathbb{Z}^d}$ of independent real Brownian motions defined on a probability space $(\Omega, \mathcal{F}, P)$. We consider the filtration $(\mathcal{F}_t)_{t\geq 0}$ where $\mathcal{F}_t$ is the complete σ-algebra generated by the W_s^i with $i \in \mathbb{Z}^d$ and $0 \leq s \leq t$.

When M is a manifold and E a linear topological space, we denote by $C^k(M, E)$ the set of map from M to E with continuous derivatives of order k. $C(M, E)$ is the set of continuous map from M to E.

We are interested in the stochastic differential system

$$X_t^i = \zeta^i + W_t^i - \frac{1}{2}\int_0^t \sum_{k\in\mathbb{Z}^d} J(i-k) X_s^k \, ds \quad \forall i \in \mathbb{Z}^d, t \in \mathbb{R}^+ \tag{1}$$

where ζ is a random vector independent from W and J an even deterministic sequence. We would like to write it in some infinite-dimensional linear space E

$$X_t = \zeta + W_t - \frac{1}{2}\int_0^t J X_s ds, \quad t \in \mathbb{R}^+, \tag{2}$$

As usually, we have to choose a state space $E \subset \mathbb{R}^{\mathbb{Z}^d}$ such that the series in the right hand-side of (1) is convergent. It's quite natural to make the following assumptions :

1. $t \mapsto W_t \in C(\mathbb{R}^+, E)$ P-almost surely.
2. E contains each finite sequence. For $i \in \mathbb{Z}^d$, we denote by e_i the sequence for which every component vanishes except the i-th which is equal to 1.
3. For each $i \in \mathbb{Z}^d$, the canonical projection π_i is continuous from E to $\mathbb{R}$.
4. The family $(e_i)_{i\in\mathbb{Z}^d}$ is a weak Schauder basis for E, *i.e.*

$$\forall x \in E, \forall \phi \in E' \qquad \lim_{\Lambda\to\mathbb{Z}^d} \sum_{i\in\Lambda} \pi_i(x)\phi(e_i) = \phi(x) \tag{3}$$

5. J is a continuous linear operator on E, with $\pi_j(Je_i) = J(i-j)$.

We now introduce our choice for E. We fix a $p > 0$ and define

$$B_p = \{x \in \mathbb{R}^{\mathbb{Z}^d}, \quad \|x\|_{B_p} = \sup_{k\in\mathbb{Z}^d} \frac{|x_k|}{(1+|k|)^p} < +\infty\}$$

and

$$B_{p,0} = \{x \in B_p, \quad \lim_{|k|\to+\infty} \frac{|x_k|}{(1+|k|)^p} = 0\}.$$

It's easy to see that $B_{p,0}$ is a separable Banach space whose $(e_i)_{i\in\mathbb{Z}^d}$ is a Schauder basis. We will see that $B_{p,0}$ is a convenient choice for E. Let us now define

$$A_p = \{x \in \mathbb{R}^{\mathbb{Z}^d}; \|x\|_{A_p} = \sum_{n\in\mathbb{Z}^d} (1+|n|)^p |x_n| < +\infty\}$$

We will always suppose that J is even and $J \in A_p$.
For $z = (z_1, ..., z_d) \in \mathbb{C}^d$ and $n = (n_1, ..., n_d) \in \mathbb{Z}^d$, we set

$$z^n = \prod_{i=1}^{n} z_i^{n_i} \text{ and } |n| = \sum_{i=1}^{d} |n_i|$$

$$\mathbb{U} = \{z \in \mathbb{C}^d, \ \forall i \in \{1, \dots, d\} \ |z_i| = 1\}$$

We introduce $\hat{J}$, the Fourier transform or the dual function of J, defined on $\mathbb{U}$ by

$$\hat{J}(z) = \sum_{n\in\mathbb{Z}^d} J(n) z^n \tag{4}$$

Since J is summable, it is clear that $\hat{J}$ defines a continuous map on $\mathbb{U}$. Let us recall that, for two sequences $u = (u_n)_{n\in\mathbb{Z}^d}, v = (v_n)_{n\in\mathbb{Z}^d}$ such that

$$\forall n \in \mathbb{Z}^d \ \sum_{k\in\mathbb{Z}^d} |u_k v_{n-k}| < +\infty,$$

the convolution $u * v$ of u and v is defined by

$$\forall n \in \mathbb{Z}^d, \ (u * v)_n = \sum_{k\in\mathbb{Z}^d} u_k v_{n-k}.$$

We recall some results and tools (see [12] for more details).

Lemma 1. $(A_p, \|.\|_{A_p}, *)$ *is an unital commutative Banach algebra.*

Lemma 2. $\forall u, v \in A_p$

- $\forall z \in \mathbb{U} \ \widehat{u * v}(z) = \hat{u}(z)\hat{v}(z)$
- $\exp(\hat{u}) = \widehat{\exp(u)}$ *on* $\mathbb{U}$.
- u *invertible* $\iff$ $\hat{u}$ *does not vanish on* $\mathbb{U}$

1.2 Some results on Gaussian Gibbs measures

In the present paper, a probability measure on a topological space Ω endowed with its borelian algebra is simply called a measure on Ω. We denote by $\mathcal{P}(\Omega)$ the set of measures on Ω. As usually, for $x \in \Omega$, the Dirac measure at point x is denoted by δ_x.

Let us introduce the concept of Gibbs measure on $\Omega = \mathbb{R}^{\mathbb{Z}^d}$. Each $\omega \in \Omega$ can be considered as a map from $\mathbb{Z}^d$ to $\mathbb{R}$. For a finite subset Λ of $\mathbb{Z}^d$, we will denote ω_Λ its restriction to Λ. Then, when A and B are two disjoint subsets of $\mathbb{Z}^d$ and $(\omega, \eta) \in \mathbb{R}^A \times \mathbb{R}^B$, $\omega\eta$ denotes the concatenation of ω and η, that is the element $z \in \mathbb{R}^{A \cup B}$ such that

$$z_i = \begin{cases} \omega_i & \text{if } i \in A \\ \eta_i & \text{if } i \in B \end{cases}$$

For finite subset Λ of $\mathbb{Z}^d$, we suppose that a function Φ_Λ has been defined, which is measurable with respect to the σ-field generated by $\{\pi_i, i \in \Lambda\}$. A collection $(\Phi_\Lambda)_\Lambda$ of such functions is said to be an potential. For a finite subset Λ of $\mathbb{Z}^d$, the quantity

$$H_\Lambda = \sum_{B:\ B \cap \Lambda \neq \emptyset} \Phi_B$$

is called the Hamiltonian on the volume Λ. Usually, H_Λ can be defined only on a subset of $\mathbb{R}^{\mathbb{Z}^d}$. We suppose that there exists a subset $\tilde{\Omega}$ of Ω such that

$$\forall \text{ finite } \Lambda \ \forall \omega \in \tilde{\Omega} \qquad \sum_{B:\ B \cap \Lambda \neq \emptyset} |\Phi_B(\omega)| < +\infty$$

We now define the so-called partition function Z_Λ : denoting by λ the Lebesgue's measure on the real line, we set

$$Z_\Lambda(\omega) = \int_{\mathbb{R}^\Lambda} \exp(-H_\Lambda(\eta_\Lambda \omega_{\Lambda^c})) d\lambda^{\otimes \Lambda}(\eta_\Lambda)$$

By convention, we set $\exp(-H_\Lambda(\eta_\Lambda \omega_{\Lambda^c})) = 0$ when the Hamiltonian is not defined. We suppose that for each ω in $\tilde{\Omega}$, we have $0 < Z_\Lambda(\omega) < +\infty$. Then, we can define for each bounded measurable function f on Ω and for each $\omega \in \tilde{\Omega}$,

$$\Pi_\Lambda f(\omega) = \frac{\int_{\mathbb{R}^\Lambda} \exp(-H_\Lambda(\eta_\Lambda \omega_{\Lambda^c})) f(\eta_\Lambda \omega_{\Lambda^c}) d\lambda^{\otimes \Lambda}(\eta_\Lambda)}{Z_\Lambda(\omega)}.$$

If a measure μ on Ω is such that $\mu(\tilde{\Omega}) = 1$, we say that μ is a Gibbs measure associated to the potential Φ when, for each bounded measurable function f and each finite subset Λ of $\mathbb{Z}^d$, we have

$$\mathbb{E}_\mu(f|(X_i)_{i\in\Lambda^c}) = \Pi_\Lambda f \quad \mu \text{ a.s.}$$

The random dynamics introduced in (1) is related with the Gibbs measures associated to the potential Φ^J defined on Ω by

$$\Phi_\Lambda^J(\omega) = \begin{cases} \frac{1}{2}J(0)\omega_i^2 & \text{if } \Lambda = \{i\} \\ J(i-j)\omega_i\omega_j & \text{if } \Lambda = \{i,j\}, i \neq j \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

Then, the corresponding Hamiltonian function is equal to

$$H_\Lambda(\omega) = \frac{1}{2}\sum_{i\in\Lambda}\sum_{j\in\Lambda} J(i-j)\omega_i\omega_j + \sum_{i\in\Lambda, j\in\Lambda^c} J(i-j)\omega_i\omega_j. \tag{6}$$

It is clear that H_Λ is well defined on $\tilde{\Omega}$, if we choose :

$$\tilde{\Omega} = \{\omega \in \mathbb{R}^{\mathbb{Z}^d}; \quad \forall i \in \mathbb{Z}^d, \quad \sum_{j\in\mathbb{Z}^d} |J(i-j)\omega_j| < +\infty\}.$$

We denote by $\mathfrak{G}_J$ the set of Gibbs measures on $\mathbb{R}^{\mathbb{Z}^d}$ associated to the potential given in (5). If $\mathfrak{G}_J$ contains more than one point, we say that phase transition occurs. As each set of Gibbs measures, $\mathfrak{G}_J$ is a convex set whose extreme points are called pure phases. (For general results on Gibbs measures, see [13].) Dobrushin and Kunsch gave the following description of $\mathfrak{G}_J$:

Proposition 1. *$\mathfrak{G}_J$ contains at least one element if and only if the following conditions are fulfilled*

1. $\hat{J}(\mathbb{U}) \subset \mathbb{R}^+$
2. $\int_\mathbb{U} \frac{1}{\hat{J}(z)} dz < \infty$*, where dz is the normalized Haar measure on the torus $\mathbb{U}$.*

In this case, $\mathfrak{G}_J$ is described as follows :

$$\mathfrak{G}_J = \{\mu_\infty * m; \quad m \in \mathcal{P}(\Omega) \text{ and } m(M_0^J) = 1\}, \tag{7}$$

where μ_∞ is the centered Gaussian measure with $\frac{1}{\hat{J}}$ as spectral density and

$$M_0^J = \{\omega \in \tilde{\Omega}; \quad \forall i \in \mathbb{Z}^d, \quad \sum_{j\in\mathbb{Z}^d} J(i-j)\omega_j = 0\} \tag{8}$$

The pure phases are obtained when m is reduced to a Dirac measure.

Remark 1. *The choice of the space state is not unique and is itself an element in the definition of Gibbs measure. By chance, we have here* $B_{p,0} \subset \tilde{\Omega}$ *and it comes immediately from the definition that* $B_{p,0}$*-Gibbs measures are exactly* $\tilde{\Omega}$*-Gibbs measure with their support in* $B_{p,0}$*. Moreover, let* $\mu \in \mathcal{P}(B_{p,0})$ *be a Gibbs measure for* J *: it can be written if the form* $\mu = \mu_\infty * m$*, with* $m(M_0^J) = 1$*. By the lemma of Borel-Cantelli, it is not difficult to see that* $\mu_\infty(B_{p,0}) = 1$*. It follows that* $m(B_{p,0}) = 1$*. In other words,*

$$\mathfrak{G}_J \cap \mathcal{P}(B_{p,0}) = \{\mu_\infty * m; \quad m \in \mathcal{P}(B_{p,0}) \text{ and } m(\ker J) = 1\}, \tag{9}$$

with

$$\ker J = B_{p,0} \cap M_0^J.$$

Remark 2. *In a previous paper [12] , we have shown the following result :*

Proposition 2. *Let* $p > 0$ *and* $J \in A_p$*.*
$\mathfrak{G}_J \cap \mathcal{P}(B_p)$ *contains exactly one element if and only if* $\hat{J} > 0$ *on* $\mathbb{U}$*.*
Then, $\mathfrak{G}_J \cap \mathcal{P}(B_p) = \{\mu_\infty\}$*.*

1.3 Existence of the infinite-dimensional diffusion and first properties

The next lemma shows some properties of J as operator.

Lemma 3. *Let* J *be an even sequence in* A_p*. Then* J *induces an operator* $\tilde{J}$ *on* B_p *such that* $\pi_j(\tilde{J}e_i) = J(i-j)$*, which maps* $B_{p,0}$ *into itself and satisfies*

$$\|\tilde{J}\|_{L(B_p)} = \|\tilde{J}\|_{L(B_{p,0})} = \|J\|_{A_p}.$$

Démonstration. The proof is classic. The key point is the inequality
$(1+|i+j|)^p \leq (1+|i|)^p(1+|j|)^p$. □

Remark 3. *This justifies the notation* $\ker J = \{u \in B_{p,0},\ Ju = 0\}$.

Lemma 4. *For* P*-almost every* ω*,* $t \mapsto W_t \in C(\mathbb{R}^+, B_{p,0})$ *holds .*

Démonstration. Let Λ_n be an increasing sequence of finite subsets of $\mathbb{Z}^d$ such that $\lim\limits_{n\to\infty} \uparrow \Lambda_n = \mathbb{Z}^d$. We denote by ${}^nW$ the process defined by ${}^nW^i = W^i$ if $i \in \Lambda_n$, 0 else. ${}^nW_t$ takes its values in a finite dimensional space, and has almost-surely continuous coordinates. Then, $t \mapsto^n W_t \in C(\mathbb{R}^+, B_{p,0})$ P-a.s. for each n. It suffices to prove that for almost every ω, ${}^nW_t(\omega)$ converges uniformly to $W_t(\omega)$ on every compact subset of $\mathbb{R}_+$. We will show that $P(\cap_{p\in\mathbb{N}, q\in\mathbb{Q}_*^+} R_{s,q}) = 1$, where

$$R_{s,q} = \{\omega : \overline{\lim}_{n\to\infty} \sup_{t\in[0,s]} \|{}^nW_t - W_t\| \leq q\}.$$

But $\sup_{t\in[0,s]} \|{}^nW_t - W_t\| = \sup_{t\in[0,s]} \sup_{i\notin\Lambda_n} \frac{|W_t^i|}{(1+|i|)^p} = \sup_{i\notin\Lambda_n} \frac{S_s^i}{(1+|i|)^p}$, with $S_s^i = \sup_{t\in[0,s]} |W_t^i|$. Thus, the complement of $R_{s,q}$ is

$$C_{p,q} = \{\omega : S_s^i > q(1+|i|)^p \text{ i.o.}\}.$$

Now, using the famous inequality $P(S_s^i > q(1+|i|)^p) \leq 2\exp(-\frac{(q(1+|i|)^p)^2}{2s})$ and the fact that $(1+|i|)^p \gg \sqrt{\ln i}$, the result follows from Borel-Cantelli's lemma. □

Remark 4. *The same arguments prove that every centered Gaussian sequence $(X_n)_{n\in\mathbb{Z}^d}$ with bounded variances belongs to $\mathcal{P}(B_{p,0})$. Particularly, $\mu_\infty \in \mathcal{P}(B_{p,0})$.*

We need the next lemma about a deterministic equation to prove the existence and uniqueness of the solution of the s.d.e. (1).

Lemma 5. *Let E be a Banach space, $w \in C(\mathbb{R}^+, E)$, J a continuous operator in E and $\zeta \in E$. Then, the integral equation*

$$x(t) = \zeta + w(t) - \int_0^t \frac{J}{2} x(s)\, ds$$

admits an unique solution $x \in C(\mathbb{R}^+, E)$: it is given by

$$x(t) = \exp(-t\frac{J}{2})\zeta + w(t) - \frac{J}{2}\int_0^t \exp(-(t-s)\frac{J}{2}) w(s)\, ds$$

Démonstration. Existence and uniqueness of a solution follows from Picard's fixed point theorem on $C([0,T], E)$.

□

Notation : If f is a borelian measurable map from Ω into Ω' and μ a measure on Ω, the distribution of f under μ is the pushforward measure $f\mu$ – also denoted by μ_f, specially when Ω is an abstract space and when f is said to be a random variable – defined on Ω' by $f\mu(A) = \mu(f^{-1}(A))$, for each borelian set A.

Theorem 1. *Let $(\Omega, \mathcal{F}, P)$ be a probability space, $(W^i)_{i\in\mathbb{Z}^d}$ a family of P-independent Brownian motions and $J \in A_p$. Then, for each $B_{p,0}$-valued random variable ζ independent of $(W^i)_{i\in\mathbb{Z}^d}$, there exists a unique $B_{p,0}$- valued process X_t with continuous paths which is solution to the diffusion equation (1) . It is given by*

$$X_t(\omega) = e^{-t\frac{J}{2}}\zeta(\omega) + W_t(\omega) - \frac{J}{2}\int_0^t e^{-(t-s)\frac{J}{2}} W_s(\omega)\, ds \tag{10}$$

X_t is a $(\mathcal{F}_t)$-adapted Markov process. For an initial law ν, the law of the process at time t is given by the following kernel :

$$T_t\nu = (\exp(-\frac{t}{2}J)\nu) * \mu_t, \quad t \geq 0, \tag{11}$$

where μ_t denotes the law at time t of the solution with 0 as initial condition.

Démonstration. Let $\Omega_1 = \{\omega : t \mapsto W_t(\omega) \in C(\mathbb{R}^+, E)\}$. Let us define, for $\omega \in \Omega_1$

$$X_t(\omega) = e^{-t\frac{J}{2}}\zeta(\omega) + W_t(\omega) - \frac{J}{2}\int_0^t e^{-(t-s)\frac{J}{2}} W_s(\omega)\ ds$$

and $X_t(\omega) = 0$ for $\omega \in \Omega \backslash \Omega_1$, the process (X_t) is well defined, and by lemma 5, it follows that X_t verifies (on Ω_1, then a.s.) the smoothness conditions and the integral equation it was asked for. Now, let (X') be another solution : we set

$$\Omega_2 = \{\omega : t \mapsto X'_t(\omega) \in C(\mathbb{R}^+, E) \text{and } X_t = \zeta + W_t - \int_0^t \frac{J}{2} X_s ds \quad \forall t \in \mathbb{R}^+\}.$$

By the uniqueness part in lemma 5, $t \mapsto X_t(\omega)$ and $t \mapsto X'(\omega)$ coincide for $\omega \in \Omega_1 \cap \Omega_2$. Since $P(\Omega_1 \cap \Omega_2) = 1$, these are indistinguishable processes. □

Definition : Let X and Y be E-valued random variables whose norm admit a second moment. We define the covariance $C_{X,Y}$ of X and Y as a continuous linear quadratic form on $E' \times E'$, where E' is endowed with the $*$-weak topology.

$$(\phi, \psi) \in E' \times E' \mapsto C_{X,Y}(\phi, \psi) = \mathbb{E}\phi(X)\psi(Y).$$

Lemma 6. *For each $q \geq 1$, there exists a positive constant M_q such that for each $\sigma \geq 0$ and every sequence $X = (X_n)_{n \in \mathbb{Z}^d}$ of centered Gaussian random variables satisfying to*

$$\forall n \in \mathbb{Z}^d \ \mathbb{E}X_n^2 \leq \sigma^2,$$

it comes

$$\mathbb{E}\|X\|_{B_p}^q \leq M_q \sigma^q. \tag{12}$$

Démonstration.

$$\begin{aligned} \mathbb{E}\|X\|_{B_p}^q &= \int_0^{+\infty} q\ t^{q-1} P(\|X\|_{B_p} > t)\ dt. \\ &\leq \sigma^q (1 + \int_1^{+\infty} q\ t^{q-1} P(\|X\|_{B_p} > \sigma t)\ dt). \end{aligned}$$

$$\begin{aligned} P(\|X\|_{B_p} > \sigma t) &= P(\cup_{n\in\mathbb{Z}^d}\{|X_n| > t(1+|n|)^p\sigma\}) \\ &\leq \sum_{n\in\mathbb{Z}^d} P(|X_n| > t(1+|n|)^p\sigma). \end{aligned}$$

Since $P(|X_n| > t(1+|n|)^p\sigma) \leq \frac{2}{\sqrt{2\pi}t(1+|n|)^p}\exp(-\frac{(t(1+|n|)^p)^2}{2})$, it suffices to prove that
$\sum_{k\in\mathbb{Z}^d}\frac{1}{(1+|n|)^p}\int_1^{+\infty} t^{q-2}\exp(-\frac{(t(1+|n|)^p)^2}{2})\ dt < +\infty$.
But since $(1+|n|)^p \geq 1$,

$$\int_1^{+\infty} t^{q-2}\exp(-\frac{(t(1+|n|)^p)^2}{2})\ dt \leq \Big(\int_1^{+\infty} t^{q-2}\exp(-\frac{t^2}{4})\ dt\Big)\exp(-\frac{((1+|n|)^p)^2}{4})$$

Since $(1+|n|)^p \gg \sqrt{\ln n}$, we may conclude. □

We will note

$$\langle .,. \rangle = C_{W_1,W_1}(.,.).$$

It is easy to see that

$$\forall s,t\geq 0 \quad C_{W_s,W_t}(.,.) = \inf(s,t)\langle .,. \rangle.$$

For $J \in L(E,E)$ and $\phi \in E'$, we define $J.\phi \in E'$ by

$$\forall x \in E \quad (J.\phi)(x) = \phi(Jx).$$

It is easy to see that the operator induced by an even sequence J in A_p is symmetric with respect to $\langle , \rangle$:

$$\forall \phi,\psi \in E' \quad \langle J.\phi,\psi\rangle = \langle \phi, J.\psi\rangle.$$

Proposition 3. *Let J be an even sequence in A_p and $(X_t)_{t\geq 0}$ the solution in $B_{p,0}$ to the s.d.e. (1) with initial condition 0. Then, the covariance of X is given by*

$$\forall t\geq 0 \quad \forall \phi,\psi\in E' \quad C_{X_t,X_t}(\phi,\psi) = \langle V_t.\phi,\psi\rangle,$$

where

$$V_t = \int_0^t \exp(-sJ)ds.$$

Furthermore, X_t admits

$$\int_0^t \exp(-s\hat{J})\ ds \tag{13}$$

as spectral density.

Démonstration. Since we have the explicit formula (10), the existence of the covariance is obvious and its computation is easy. The exact computation of V_t can be found in [11]. Now, first remark that when $C_{Y,Y}(\phi, \psi) = \langle B.\phi, \psi \rangle$ and when B is the Toeplitz operator associated to the even sequence $b \in A_p$, *i.e.* with $\pi_i(Be_j) = b(i-j)$, it comes

$$\begin{aligned}
\mathbb{E}Y_iY_j &= C_{Y,Y}(\pi_i, \pi_j) = \langle B.\pi_i, \pi_j \rangle = (B.\pi_i)(e_j) \\
&= \pi_i(Be_j) = B(i,j) = b(i-j) \\
&= \int_{\mathbb{U}} \hat{b}(z) dz
\end{aligned}$$

Since $\int_0^t \widehat{\exp(-sJ)}\ ds = \int_0^t \exp(-s\hat{J})\ ds$, we can conclude. □

2 Time Asymptotics

The first step in the study of the asymptotic behaviour of the diffusion X_t solution of (1) is the study of the law μ_t of the solution of (1) with initial condition 0. When J has finite support, the asymptotic behaviour of μ_t has been studied by Deuschel in [7]. However, he studies the convergence with respect to the product topology in $\mathbb{R}^{\mathbb{Z}^d}$ but we use here the topology of $B_{p,0}$. Moreover, we are interested in the speed of convergence. Therefore, we need some tools.

2.1 Metrization of the convergence

We denote by R_2 the following set of test functions

$$R_2 = \left\{ \begin{array}{l} f \in C^2(B_{p,0}, \mathbb{R}) \\ \sup_{x \in B_{p,0}} |f(x)| + \sup_{x \in B_{p,0}} \|D_x f\| + \sup_{x \in B_{p,0}} \|D_x^2 f\| \leq 1 \end{array} \right\}$$

where we set

$$\|D_x^k\| = \sup\{|D_x^k(h_1 \otimes \cdots \otimes h_k); \|h_i\|_{B_{p,0}} \leq 1, \forall i \in \{1, \ldots, k\}\}$$

Finally, we define for each $\mu, \nu \in \mathcal{P}(B_{p,0})$:

$$d(\mu, \nu) = \sup_{f \in R_2} |\int f\ d\mu - \int f\ d\nu| \tag{14}$$

Since $B_{p,0}$ is separable, the distance d metrizes the weak convergence in $\mathcal{P}(B_{p,0})$.

Lemma 7.

$$\forall \mu_1, \mu_2, \nu_1, \nu_2 \in \mathcal{P}(B_{p,0}) \quad d(\mu_1 * \mu_2, \nu_1 * \nu_2) \leq d(\mu_1, \nu_1) + d(\mu_2, \nu_2) \tag{15}$$

Démonstration. See [2], page 37. □

Lemma 8. *Let μ_1, μ_2 be Gaussian centered measures on B_p admitting ϕ_1 and ϕ_2 as spectral density. Then*

$$d(\mu_1, \mu_2) \leq M_2 \|\phi_1 - \phi_2\|_1,$$

where

$$\|\phi_1 - \phi_2\|_1 = \int_{\mathbb{U}} |\phi_1(z) - \phi_2(z)| \; dz$$

and M_2 is the constant appearing in (12) for $q = 2$.

Démonstration. Let (Ω, F, P) be a space with a probability measure supporting independent centered random variables X, Y, Z, T such that $P_X = \mu_1$, $P_Y = \mu_2$ ant that the law of Z (*resp.* of T) under P is the centered Gaussian law with $(\phi_1 - \phi_2)^-$ (resp. $(\phi_1 - \phi_2)^+$ as spectral density. Let $f \in R_2$. We have

$$\begin{aligned}
\int f \; d\mu_1 - \int f \; d\mu_2 &= \mathbb{E}f(X) - \mathbb{E}f(Y) \\
&= (\mathbb{E}f(X) - \mathbb{E}f(X+Z)) \\
&\quad + \ (\mathbb{E}f(X+Z) - \mathbb{E}f(T+Y)) \\
&\quad + \ (\mathbb{E}f(T+Y) - \mathbb{E}f(Y))
\end{aligned}$$

But $X + Z$ and $T + Y$ are Gaussian, centered, and have the same spectral density $\phi_1 + (\phi_1 - \phi_2)^- = (\phi_1 - \phi_2)^+ + \phi_2$: then, they have the same law. In particular, $\mathbb{E}f(X + Z) = \mathbb{E}f(T + Y)$, and then

$$\mathbb{E}f(X) - \mathbb{E}f(Y) = (\mathbb{E}f(X) - \mathbb{E}f(X+Z)) + (\mathbb{E}f(T+Y) - \mathbb{E}f(Y))$$

By Taylor's formula in Banach spaces, (see for example [3], p. 77),

$$f(T+Y) - f(Y) = Df_Y(T) + \int_0^1 (1-t) D^2 f_{Y+tT}(T \otimes T) \; dt$$

Hence

$$\mathbb{E}f(Y+T) - \mathbb{E}f(Y) = \mathbb{E}Df_Y(T) + \mathbb{E}\int_0^1 (1-t) D^2 f_{Y+tT}(T \otimes T) \; dt$$

Since Df is bounded and since Y and T are independent variables, we have

$$\mathbb{E}Df_Y(T) = \mathbb{E}Df_Y(\mathbb{E}T) = \mathbb{E}Df_Y(0) = 0.$$

Moreover, $\|D^2 f\| \leq 1$, so that

$$|D^2 f_{Y+tT}(T \otimes T)| \leq \|T\|^2$$

and we obtain

$$|\mathbb{E}f(T+Y) - \mathbb{E}f(Y)| \leq \frac{1}{2}\mathbb{E}\|T\|^2.$$

By the same way $|\mathbb{E}f(X+Z) - \mathbb{E}f(X)| \leq \frac{1}{2}\mathbb{E}\|Z\|^2$. By lemma 6, we get $\mathbb{E}(\|Z\|^2_{B_p}) \leq M_2\|(\phi_1 - \phi_2)^-\|_1$
and $\mathbb{E}(\|T\|^2_{B_p}) \leq M_2\|(\phi_1 - \phi_2)^+\|_1$. Now

$$|\mathbb{E}f(X) - \mathbb{E}f(Y)| \leq M_2\|(\phi_1 - \phi_2)^-\|_1 + M_2\|(\phi_1 - \phi_2)^+\|_1 = M_2\|\phi_1 - \phi_2\|_1$$

We conclude that $d(\mu_1, \mu_2) \leq M_2\|\phi_1 - \phi_2\|_1$. □

2.2 Asymptotics with zero as initial condition

We now look at the asymptotic behaviour of μ_t.

Theorem 2. *The following assertions are equivalent :*

1. *(μ_t) has a limit point when t tends to $+\infty$.*
2. *$\underline{\lim}_{t\to+\infty} \int_{B_{p,0}} (\pi_0(x))^2 \, d\mu_t(x) < +\infty$.*
3. *$\mathfrak{G}_J$ is not empty.*
4. *(μ_t) converges in $\mathcal{P}(B_{p,0})$ when t tends to $+\infty$.*

When one of these conditions is fullfilled, μ_t converges to μ_∞, the Gaussian measure on $\mathbb{R}^{\mathbb{Z}^d}$ with $\frac{1}{\hat{J}}$ as spectral density.

Démonstration. 1. ⇒ 2. If (μ_t) has a limit point when t tends to $+\infty$, so does $(\pi_0\mu_t)$. But a convergent subsequence of Gaussian r.v. has bounded variances. Hence, 2. follows.
2. ⇒ 3.

$$\int (\pi_0(x))^2 \, d\mu_t(x) = \int_0^t \int_{\mathbb{U}} e^{-t\hat{J}(z)} \, dz \, dt \geq t\chi(\hat{J} \leq 0),$$

when χ is the normalized Haar measure on $\mathbb{U}$. Then, $\hat{J}$ is a.e. positive, and – since it is continuous – always nonnegative. Since $\hat{J}$ is a.e. positive,

$\frac{1-\exp(-t\hat{J}(z))}{\hat{J}(z)}$ is a nice expression for the spectral density of μ_t. Now, by Fatou's lemma

$$\int_{\mathbb{U}} \frac{1}{\hat{J}(z)} dz \le \underline{\lim}_{t\to\infty} \int_{\mathbb{U}} \frac{1-\exp(-t\hat{J}(z))}{\hat{J}(z)}\, dz = \underline{\lim}_{t\to\infty} \int (\pi_0(x))^2\, d\mu_t(x) < +\infty$$

Now, Proposition 1 implies that $\mathfrak{G}_J$ is not empty.
3. $\Rightarrow$ 4. Always by Proposition 1, $\hat{J}$ is nonnegative and $\frac{1}{\hat{J}}$ integrable. Therefore, $\hat{J}$ is a.e. positive. Since μ_t (*resp.* μ_∞)admits $\frac{1-\exp(-t\hat{J})}{\hat{J}}$ (*resp.* $\frac{1}{\hat{J}}$) as spectral density, it follows from lemma 8 that

$$d(\mu_t, \mu_\infty) \le M_2 \int_{\mathbb{U}} \frac{\exp(-t\hat{J}(z))}{\hat{J}(z)}\, dz$$

The right-handside tends to zero by dominated convergence.
4. $\Rightarrow$ 1. Obvious. □

From now on, we will suppose that assumptions of Theorem 2 are fullfilled.

2.3 Exponential bounds in A_p

Lemma 9. *Let $k \in \mathbb{Z}_+$ et $f \in C^k(\mathbb{U}, \mathbb{C})$. For each $n \in \mathbb{Z}_+^d$ such that $|n| \le k$, there exists a function $g_n \in C^\infty(\mathbb{R}^+ \times \mathbb{U}, \mathbb{C})$ such that*

$$\partial_n \exp(-tf) = \exp(-tf) g_n(t, .)$$

and

$$\|g_n(t, .)\|_\infty = O(t^{|n|}).$$

Démonstration. We get the proof by natural induction on $|n|$ with the help of the multivariate Leibnitz formula. □

Notation : For $f \in C^N(\mathbb{U}, \mathbb{C})$, we define

$$\|f\|_{D^N} = \max\{|\partial_k f(x)|; x \in \mathbb{U}, |k| \le N\}.$$

Lemma 10. *Let $k \in \mathbb{Z}_+$ such that $2k > \frac{d}{2} + p$; then*

$$\forall f \in C^{2k}(\mathbb{U}, \mathbb{C}) \quad \exists! F \in A_p \quad f = \hat{F}.$$

Moreover, there exists a constant K such that

$$\|F\|_{A_p} \le K \|(1-\Delta)^k f\|_{L^2(\mathbb{U})}.$$

Démonstration. If such a sequence F exists, it necessary satisfies to $F_n = \langle f, \chi_n \rangle$, where $\chi_n(z) = z^n$. Then, let us define F in such a way.

$$\begin{aligned} F_n = \langle f, \chi_n \rangle &= \frac{1}{(1+|n|^2)^k} \langle f, (1+|n|^2)^k \chi_n \rangle \\ &= \frac{1}{(1+|n|^2)^k} \langle f, (1-\Delta)^k \chi_n \rangle \\ &= \frac{1}{(1+|n|^2)^k} \langle (1-\Delta)^k f, \chi_n \rangle \end{aligned}$$

Hence $(1+|n|)^p F_n = \frac{(1+|n|)^p}{(1+|n|^2)^k} \langle (1-\Delta)^k f, \chi_n \rangle$, and next, by Cauchy-Schwarz :

$$\|F\|_{A_p} \leq \Big(\sum_{n \in \mathbb{Z}^d} \frac{(1+|n|^p)^2}{(1+|n|^2)^{2k}} \Big)^{\frac{1}{2}} \|(1-\Delta)^k f\|_{L^2(\mathbb{U})}$$

□

Lemma 11. *Let $\phi \in A_p$ such that $\hat{\phi} \in C^{2k}(\mathbb{U}, \mathbb{C})$, where k is an integer such that $2k > \frac{d}{2} + p$. Then*

1. $\forall F \in A_p, \quad \forall \varepsilon > 0$, *we have*

$$\|\phi * \exp(-tF)\|_{A_p} = o(e^{-(m_\phi(F)-\varepsilon)t}),$$

where

$$m_\phi(F) = \inf\{Re\ \hat{F}(z); z \in \text{supp}\ \hat{\phi}\}.$$

2. *When $\hat{F} \in C^{2k}(\mathbb{U}, \mathbb{C})$, we can be more precise : there exists a constant $K_{F,k,p}$ independent from ϕ such that*

$$\forall t \geq 0 \quad \|\phi * \exp(-tF)\|_{A_p} \leq K_{F,k,p} \|\hat{\phi}\|_{D^{2k}} t^{2k} e^{-m_\phi(F)t}.$$

Démonstration. We first prove the second part of the lemma : let F be such that $\hat{F} \in C^{2k}(\mathbb{U}, \mathbb{C})$. By lemma 10, we have

$$\begin{aligned} \|\phi * \exp(-tF_n)\|_{A_p} &\leq K \|(1-\Delta)^k (\hat{\phi} \exp(-t\hat{F}_n))\|_{L^2(\mathbb{U})} \\ &\leq K \sup\{|(1-\Delta)^k (\hat{\phi} \exp(-t\hat{F}_n))(z)|, z \in \text{supp}\ \hat{\phi}\} \end{aligned}$$

If we expand $(1-\Delta)^k$ by Newton's formula, and next by Leibnitz's, we obtain the existence of a constant $K^{(1)}$ such that

$$\forall f, g \in C^{2k}(\mathbb{U}, \mathbb{C})\ \|(1-\Delta)^k fg\|_\infty \leq K^{(1)} \|f\|_{D^{2k}} \max\{|\partial_i g(x)|; x \in \text{supp}\ f, |i| \leq 2k\}.$$

Thus, we get

$$\begin{aligned}\|\phi * \exp(-tF_n)\|_{A_p} &\leq KK^{(1)} \max_{|i|\leq 2k} \|g_i(t,.)\|_\infty \|\hat{\phi}\|_{D^{2k}} (e^{-m_\phi(F)t}) \\ &\leq K_{F,k,p} \|\hat{\phi}\|_{D^{2k}} t^{2k} e^{-m_\phi(F)t},\end{aligned}$$

The last inequality comes from lemma 9, and the step 2. is done..

Next go to the general case : let $F \in A_p$. For $n \in \mathbb{Z}_+$, set $F_n(i) = F(i)$ if $|i| \leq n$, 0 else. Obviously, $\hat{F_n}$ is C^∞ and F_n tends to F. Let us choose n such that $\|F - F_n\|_{A_p} < \varepsilon/3$. We have

$$|m_\phi(F_n) - m_\phi(F)| \leq \|\hat{F_n} - \hat{F}\|_\infty \leq \|F_n - F\|_{A_p} < \frac{\varepsilon}{3}$$

and

$$\phi * \exp(-tF) = \phi * \exp(-tF_n) * \exp(t(F_n - F))$$

Hence

$$\begin{aligned}\|\phi * \exp(-tF)\|_{A_p} &\leq \|\phi * \exp(-tF_n)\|_{A_p} \| \exp(t(F_n - F))\|_{A_p} \\ &\leq \|\phi * \exp(-tF_n)\|_{A_p} e^{\|F_n - F\|_{A_p} t}\end{aligned}$$

Now,

$$\|\phi * \exp(-tF)\|_{A_p} \leq K_{F_n,k,p} \|\hat{\phi}\|_{D^{2k}} t^{2k} e^{-(m_\phi(F) - \frac{2}{3}\varepsilon)t},$$

which achieves the proof. □

2.4 Spectral analysis

Definitions : For $u \in B_p$, we define $\tilde{u} \in A_p'$ by

$$\forall x \in A_p \quad \tilde{u}(x) = \sum_{n \in \mathbb{Z}^d} u_n x_n,$$

and we call *spectral support* of the sequence u the following subset of $\mathbb{U}$:

$$\text{spec } u = \text{supp } \tilde{u} = \complement \cup_{O \in \mathcal{O}_u} O,$$

where

$$\mathcal{O}_u = \{O \text{ open set in } \mathbb{U} \quad , \quad \forall f \in A_p \text{ supp } \hat{f} \subset O \Rightarrow \tilde{u}(f) = 0\}.$$

It is easy to check that

$$\forall u \in B_p, \forall \lambda \in \mathbb{C} - \{0\} \quad \text{spec } (\lambda u) = \text{spec } u$$

and

$$\forall u, v \in B_p \quad \text{spec } (u+v) \subset \text{spec } u \cup \text{spec } v$$

Fundamental example : Let u be defined by $u_n = z^n$, with a fixed $z \in \mathbb{U}$. We will prove that spec $u = \{z\}$. First show $z \in$ spec u. Let O be an open set containing z. We can find f of class C^∞, such that $f(z) = 1$ and whose support is a subset of O. Now, we get $J \in A_p$ such that $f = \hat{J}$:

$$\tilde{u}(J) = \sum_{n \in \mathbb{Z}^d} J(n) z^n = \hat{J}(z) = f(z) = 1,$$

which proves that $O \notin \mathcal{O}_u$. Then, no element in $\mathcal{O}_u$ can contain z *i.e.* $z \in$ spec u. Conversely, let $x \in \mathbb{U} \backslash \{z\}$. We can find a neighbourhood O of x which does not contain z. Now, let $J \in A_p$ such that supp $\hat{J} \subset O$: thus $\tilde{u}(J) = \hat{J}(z) = 0$, since z is not in O, *a fortiori* not in the support of $\hat{J}$. This proves that $O \in \mathcal{O}_u$, and hence $x \notin$ spec u, since $x \in O$.

Lemma 12.

$u \mapsto \tilde{u}$ isometrically maps B_p into A'_p.

Démonstration. It is equivalent to the well-known fact that the set of bounded sequences is the dual of the set of absolutely convergent series. □

Lemma 13.

For an even sequence $A \in A_p$ and $u \in B_p$, we have $\widetilde{Au} = A.\tilde{u}$

Lemma 14. *For each $u \in B_p$, $\mathcal{O}_u$ is stable under union.*

Démonstration. Since the proof is well-known in the context of distributions, we will omit it. The key is that the Banach algebra of functions $\hat{A_p}$ allows to separate a point from a compact set, and then we can exhibit a partition of unity. Such Banach algebras are said to be regulars (see [17], chap VIII). □

Lemma 15. *Let $f, g \in A_p$ and $\tilde{u} \in A'_p$. If $\hat{f}$ and $\hat{g}$ coincide on a neighborhood of supp $\tilde{u}$, then $\tilde{u}(f) = \tilde{u}(g)$.*

Démonstration. By linearity, we may assume $g = 0$. Let V be an open set containing supp $\tilde{u}$ on which $\hat{J}$ identically vanishes : then

$$\text{supp } \hat{f} \subset \complement V \subset \cup_{O \in \mathcal{O}_u} O = O'.$$

By lemma 14, $O' \in \mathcal{O}_u$, then, by definition of $\mathcal{O}_u$, we get $\tilde{u}(f) = 0$. □

2.5 Asymptotics for nonzero initial condition

Notations : We denote by Υ_J the map from $B_{p,0}$ to $[0,+\infty]$ defined by

$$\Upsilon_J(u) = \inf\{\hat{J}(z); z \in \text{spec } u\}.$$

For $\tau > 0$, we set

$$E_{J,\tau} = \{v \in B_{p,0}; \quad \Upsilon_J(v) \in [\tau, +\infty]\}.$$

We can now claim the main theorem of this section.

Theorem 3. *Let J be a potential in A_p. For $x \in B_{p,0}$, let us denote by μ_t^x the law at time t of the s.d.e. (1) with initial condition $\zeta = x$.*

1. Asymptotic behaviour
If x has the following decomposition $x = u + s$ with $s \in \ker J$ and $\inf\{\hat{J}(z); z \in \text{spec } u\} > 0$*, then we have*

$$\mu_t^x \underset{t\to+\infty}{\Rightarrow} \tau_s \mu_\infty,$$

where $\tau_s \mu_\infty$ is the image of μ_∞ by the translation τ_s by vector s.

2. Speed of convergence in case of uniqueness of the Gibbs measure
If $\hat{J}$ is strictly positive, there is uniqueness of the Gibbs measure. Let $a = \inf_{\mathbb{U}} \hat{J} > 0$. Then, for each $b < a$, we can find a constant L_b such that

$$\forall x \in B_{p,0} \; \forall t \geq 0 \quad d(\mu_t^x, \mu) \leq \frac{M_2}{a} e^{-at} + L_b \|x\|_{B_p} e^{-\frac{b}{2}t},$$

where M_2 has been defined in lemma 8.

3. Speed of convergence in case of phase transition
We suppose that

$$\{z \in \mathbb{U}; \hat{J}(z) = 0\} = \{(1, \dots, 1)\}$$

and that there exists $A \in Gl_d(\mathbb{R})$ and $d_0 < d$ such that the following equivalence holds at point 0 :

$$f(x) \sim \|Ax\|^{d_0},$$

where

$$f(\theta_1, \dots, \theta_d) = \hat{J}(e^{i\theta_1}, \dots, e^{i\theta_d}).$$

Then, there exists constants K and L such that, if $x \in B_{a,0}$ satisfies $x = u + s$ with $s \in \ker\ J$ and $\inf\{\hat{J}(z); z \in spec\ u\} > 0$, we have

$$K \leq \underline{\lim}_{t\to\infty} d(\mu_t^x, \tau_s \mu) t^{\frac{d}{d_0}-1} \leq \overline{\lim}_{t\to\infty} d(\mu_t^x, \tau_s \mu) t^{\frac{d}{d_0}-1} \leq L. \tag{16}$$

Let us first make some remarks.

Remark 5. *The set of fastly decreasing sequences can be written in the form* $\mathcal{S}(\mathbb{Z}^d) = \cap_{p\in\mathbb{Z}_+} A_p$ *: the family of norms* $(\|.\|_{A_p})_{p\in\mathbb{Z}_+}$ *induces the usual Frechet space topology on* $\mathcal{S}(\mathbb{Z}^d)$*. Conversely, the set of slowly increasing sequence can be written* $\mathcal{S}'(\mathbb{Z}^d) = \cup_{p\in\mathbb{Z}_+} B_{p,0}$ *and the family of norms* $(\|.\|_{B_{p,0}})_{p\in\mathbb{Z}_+}$ *induces its usual topology as Frechet space. Therefore, Theorem 3 can be reformulated with* $J \in \mathcal{S}(\mathbb{Z}^d)$ *and* $x \in \mathcal{S}'(\mathbb{Z}^d)$*, but the initial form is of course more precise.*

Démonstration. The main step is contained in the following lemma.

Lemma 16. *1. Let* $u \in B_{p,0}$ *admitting the decomposition* $u = v + s$ *with* $s \in \ker J$*. Then, for each* $\varepsilon > 0$ *:*

$$\|\exp(-tJ)u - s\|_{B_p} = o(e^{-(\Upsilon_J(v)-\varepsilon)t}).$$

2. If $\hat{J} \in C^{2k}(\mathbb{U}, \mathbb{C})$*, with* $2k > \frac{d}{2} + p$*, we can be more precise : there exists a constant* $K'_{J,k,p}$ *such that*

$$\begin{aligned} &\forall u \in B_{p,0}, u = v + s,\ \ s \in \ker J \\ &\qquad \forall t \geq 1\ \|\exp(-tJ)u - s\|_{B_p} \ \ \leq\ \ K'_{J,k,p}\|v\|_{B_p} t^{4k} e^{-\Upsilon_J(v)t}. \end{aligned}$$

Démonstration. At first, $\exp(-tJ)u = \exp(-tJ)v + \exp(-tJ)s$. But since $Js = 0$, we have $\exp(-tJ)s = s$ and it just remains to control $\exp(-tJ)v$. Now

$$\|\exp(-tJ)v\|_{B_p} = \|\exp(-tJ).\tilde{v}\|_{A'_p} = \sup_{\|\phi\|_{A_p}\leq 1} |\tilde{v}(\exp(-tJ) * \phi)|$$

Let $\hat{\psi}$ be a C^{2k}-smooth function with $2k > \frac{d}{2} + p$ which is equal to 1 on

$$\{z \in \mathbb{U} \quad \hat{J}(z) \geq \Upsilon_J(v) - \frac{\varepsilon}{4}\}$$

and vanishes on

$$\{z \in \mathbb{U} \quad \hat{J}(z) \leq \Upsilon_J(v) - \frac{\varepsilon}{2}\}.$$

By lemma 10, $\hat{\psi}$ is the Fourier transform of a sequence $\psi \in A_p$. We have $\operatorname{supp} \tilde{v} \subset \{z \in \mathbb{U}; \quad \hat{J}(z) \geq \Upsilon_J(v)\}$.
Since $\hat{J}$ is continuous, $\{z \in \mathbb{U} \quad \hat{J}(z) \geq \Upsilon_J(v) - \frac{\varepsilon}{4}\}$ is a neighborhood of $\operatorname{supp} \tilde{v}$ and it follows by lemma 15 that

$$\tilde{v}(\exp(-tJ) * \phi) = \tilde{v}(\psi * \exp(-tJ) * \phi)$$

Now

$$|\tilde{v}(\psi * \exp(-tJ) * \phi)| \leq \|\tilde{v}\|_{A'_p} \|\psi * \exp(-tJ)\|_{A_p} \|\phi\|_{A_p} \tag{17}$$

Hence, we have

$$\|\exp(-tJ)v\|_{B_p} \leq \|v\|_{B_p} \|\psi * \exp(-tJ)\|_{A_p}$$

But $m_\psi(J) \geq \Upsilon_J(v) - \frac{\varepsilon}{2}$, and the first desired assertion follows from the first part of lemma 11. Now, we must be more precise to get the second desired result : we have to control the right-hand-side of (17), and, therefore, to choose ϕ and ε in a nice way.

Let Ψ be a $C^\infty(\mathbb{R}, \mathbb{R})$ function mapping $]-\infty, 0]$ to 0 and $[1, +\infty[$ to 1. For real numbers a, b such that $a < b$, we define

$$\Psi_{a,b}(x) = \Psi(\frac{x-a}{b-a}),$$

in that way that $\Psi_{a,b}$ is a $C^\infty(\mathbb{R}, \mathbb{R})$ function mapping $]-\infty, a]$ to 0 and $[b, +\infty[$ to 1. It is easy to see that, for each $i \geq 0$,

$$\sup_{\mathbb{R}} |\Psi_{a,b}^{(i)}| = \frac{1}{(b-a)^i} \sup_{\mathbb{R}} |\Psi^{(i)}|.$$

Moreover, there exists a constant M_N such that for each $g \in C^\infty(\mathbb{R}, \mathbb{R})$ and each $f \in C^N(\mathbb{U}, \mathbb{R})$, we have

$$\|g \circ f\|_{D^N} \leq M_N \|f\|_{D^N} \max_{0 \leq i \leq N} \sup_{\mathbb{R}} |g^{(i)}|.$$

We now put

$$\hat{\psi} = \Psi_{\tau - \frac{\varepsilon}{2}, \tau - \frac{\varepsilon}{4}} \circ \hat{J},$$

with the notation $\tau = \Upsilon_J(v)$. ε is not yet determined. It comes

$$\begin{aligned}
\|\hat{\psi}\|_{D^{2k}} &\leq M_{2k} \|\hat{J}\|_{D^{2k}} \max_{0 \leq i \leq 2k} \sup_{\mathbb{R}} |\Psi^{(i)}_{\tau - \frac{\varepsilon}{2}, \tau - \frac{\varepsilon}{4}}| \\
&\leq M_{2k} \|\hat{J}\|_{D^{2k}} \max_{0 \leq i \leq 2k} (\frac{4}{\varepsilon})^i \sup_{\mathbb{R}} |\Psi^{(i)}| \\
&\leq M_{2k} \|\hat{J}\|_{D^{2k}} \max(1, (\frac{4}{\varepsilon})^{2k}) \max_{0 \leq i \leq 2k} \sup_{\mathbb{R}} |\Psi^{(i)}|
\end{aligned}$$

Let $t \geq 1$ be a *fixed* number. Now, we put $\varepsilon = \frac{4}{t}$, and apply the second part of lemma 11 to control $\|\psi * \exp(-tJ)\|_{A_p}$: we have

$$m_\psi(J) \geq \Upsilon_J(v) - \frac{\varepsilon}{2} = \Upsilon_J(v) - \frac{2}{t},$$

from which follows

$$\|\exp(-tJ)u - s\|_{B_p} \leq K'_{J,k,p}\|v\|_{B_p} t^{4k} e^{-\Upsilon_J(v)t},$$

with

$$K'_{J,k,p} = e^2 K_{J,k,p} M_{2k} \|\hat{J}\|_{D^{2k}} \max_{0\leq i\leq 2k} \sup_{\mathbb{R}} |\Psi^{(i)}|.$$

□

Now, since

$$\begin{aligned} d(\mu_t^x, \tau_s\mu_\infty) &= d(\tau_s\tau_{e^{-t\frac{J}{2}}u}\mu_t, \tau_s\mu_\infty) \\ &= d(\tau_{e^{-t\frac{J}{2}}u}\mu_t, \mu_\infty) \end{aligned}$$

we get

$$|d(\mu_t^x, \tau_s\mu_\infty) - d(\mu_t, \mu_\infty)| \leq d(\tau_{e^{-t\frac{J}{2}}u}\mu_t, \mu_t) \leq \|e^{-t\frac{J}{2}}u\|_{B_p}$$

By lemma 16 and theorem 2, we get the first assertion of the lemma.
In case of uniquess, we have

$$d(\mu_t^x, \mu_\infty) \leq d(\mu_t, \mu_\infty) + \|e^{-t\frac{J}{2}}x\|_{B_p} \leq M_2 \int_{\mathbb{U}} \frac{\exp(-t\hat{J}(z))}{\hat{J}(z)}\, dz + \|e^{-t\frac{J}{2}}\|_{A_p}\|x\|_{B_p}$$

Using lemma 11 and the fact that $\hat{J} \geq a$, we get the desired speed of convergence.

In order to determinate the speed of convergence in case of phase transition, we will use a lemma in the spirit of the Laplace method.

Lemma 17. *We suppose that*

$$\{z \in \mathbb{U}; \hat{J}(z) = 0\} = \{(1, \ldots, 1)\}$$

Let $A \in Gl_d(\mathbb{R})$ and $d_0 < d$. We define a function f by :

$$f(\theta_1, \ldots, \theta_d) = \hat{J}(e^{i\theta_1}, \ldots, e^{i\theta_d}).$$

We have the following results :

1. *If*

$$\underline{\lim}_{x\to 0} f(x)\|Ax\|^{-d_0} \geq 1,$$

then

$$\overline{\lim}_{t\to\infty} \int_{\mathbb{U}} \frac{e^{-t\hat{J}(z)}}{\hat{J}(z)}\, dz\; t^{\frac{d}{d_0}-1} \leq \frac{1}{\det A} \frac{K_d}{(2\pi)^d} \Gamma(\frac{d}{d_0} - 1) \tag{18}$$

2. If

$$\overline{\lim}_{x\to 0} f(x)\|Ax\|^{-d_0} \leq 1,$$

then

$$\underline{\lim}_{t\to\infty} \int_{\mathbb{U}} \frac{e^{-t\hat{J}(z)}}{\hat{J}(z)}\ dz\ t^{\frac{d}{d_0}-1} \geq \frac{1}{\det A}\frac{K_d}{(2\pi)^d}\Gamma(\frac{d}{d_0}-1), \tag{19}$$

where the constant K_d only depends from the lattice dimension d.

Démonstration. Since the proof of (19) is similar to proof of (18), we will only prove (18). As usually $B(r) = \{x \in \mathbb{R}^d; \|x\|_2 \leq r\}$, where $\|.\|_2$ is the canonical euclidian norm in $\mathbb{R}^d$. Then, let us suppose that $\underline{\lim}_{x\to 0} f(x)\|Ax\|^{-d_0} \geq 1$. For each $\varepsilon > 0$, we can find $\alpha < 1$ such that

$$\|Ax\| \leq \alpha \Longrightarrow f(x) \geq (1-\varepsilon)\|Ax\|^{d_0}$$

Since f does not vanish on the closure of $[-\pi,\pi]^d \backslash A^{-1}\ B(\alpha)$, we have $b = \inf_{A^{-1}\ B(\alpha)} f > 0$, and

$$\int_{\mathbb{U}} \frac{e^{-t\hat{J}(z)}}{\hat{J}(z)}\ dz\ = \frac{1}{(2\pi)^d}\int_{A^{-1}\ B(\alpha)} \frac{1}{f(x)} e^{-tf(x)}\ dx + O(e^{-bt})$$

We have

$$\begin{aligned}
\frac{1}{(2\pi)^d}\int_{A^{-1}\ B(\alpha)} \frac{1}{f(x)} e^{-tf(x)}\ dx &= \frac{1}{(2\pi)^d}\frac{1}{\det A}\int_{B(\alpha)} \frac{1}{f(A^{-1}x)} e^{-tf(A^{-1}x)}\ dx \\
&\leq \frac{1}{(2\pi)^d}\frac{1}{\det A}\frac{1}{1-\varepsilon}\int_{B(\alpha)} \frac{1}{\|x\|^{d_0}} e^{-t(1-\varepsilon)\|x\|^{d_0}}\ dx \\
&\leq \frac{1}{(2\pi)^d}\frac{1}{\det A}\frac{1}{1-\varepsilon} K_d \int_0^{\alpha} r^{d-1}\ \frac{1}{r^{d_0}} e^{-t(1-\varepsilon)r^{d_0}}\ dr,
\end{aligned}$$

where K_d is the surface of the unit sphere of $\mathbb{R}^d$.

By the change of variable $u = (1-\varepsilon)r^{d_0}$, we get

$$\int_0^{\alpha} r^{d-1}\ \frac{1}{r^{d_0}} e^{-t(1-\varepsilon)r^{d_0}}\ dr = (1-\varepsilon)^{-\frac{d}{d_0}+1}\frac{1}{t^{\frac{d}{d_0}-1}}\int_0^{(1-\varepsilon)t\alpha^{d_0}} e^{-u} u^{\frac{d}{d_0}-2}\ du$$

Since $\lim_{t\to\infty}\int_0^{(1-\varepsilon)t\alpha^{d_0}} e^{-u}u^{\frac{d}{d_0}-2}\ du = \Gamma(\frac{d}{d_0}-1)$, we deduce

$$\overline{\lim}_{t\to\infty} \int_{\mathbb{U}} \frac{e^{-t\hat{J}(z)}}{\hat{J}(z)}\ dz\ t^{\frac{d}{d_0}-1} \leq \frac{1}{\det A}\Gamma(\frac{d}{d_0}-1)\frac{K_d}{(2\pi)^d}\frac{1}{(1-\varepsilon)^{\frac{d}{d_0}}}$$

Since $\varepsilon > 0$ is arbitrary,it gives

$$\overline{\lim}_{t\to\infty} \int_{\mathbb{U}} \frac{e^{-t\hat{J}(z)}}{\hat{J}(z)} \, dz \; t^{\frac{d}{d_0}-1} \leq \frac{1}{\det A}\Gamma(\frac{d}{d_0}-1)\frac{K_d}{(2\pi)^d}. \tag{20}$$

□

Now, using the same inequalities as before and the first part of lemma 17, we see that we can take for the constant in (16) $K = M_2\frac{1}{\det A}\frac{K_{d+1}}{2\pi}\Gamma(\frac{d}{d_0}-1)$. Let us show the second inequality in (16) : by the second part of lemma 17, it suffices to prove the existence of a constant $N > 0$ such that

$$\forall t > 0 \quad d(\mu_t, \mu) \geq N\|\Phi_t - \Phi_\infty\|_1, \tag{21}$$

with $\Phi_t(z) = \frac{1}{\hat{J}(z)}(1 - e^{-t\hat{J}(z)})$ and $\Phi_\infty(z) = \frac{1}{\hat{J}(z)}$. Let $\sigma_\infty^2 = \|\Phi_\infty\|_1$ and $\sigma_t^2 = \|\Phi_t\|_1$. Since $\Phi_\infty \geq \Phi_t$, $\|\Phi_t - \Phi_\infty\|_1 = \|\Phi_\infty\|_1 - \|\Phi_t\|_1 = \sigma_\infty^2 - \sigma_t^2$. This simple remark will be important. Let $\phi \in C^\infty(\mathbb{R}, \mathbb{R}^+)$, equal to 1 for $|x| \leq \frac{1}{4}\sigma_\infty$ and vanishing for $|x| \geq \frac{1}{2}\sigma_\infty$. We define

$$g(s) = \mathbb{E}_{\mathcal{N}(0,s)}\phi = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi s}}\phi(x)e^{-\frac{x^2}{2s}} \, dx.$$

Now, we can choose $H > 0$ such that the function $f : B_{p,0} \to \mathbb{R}$ defined by

$$f(x) = Hg(\pi_0(x))$$

belongs to R_2. Now

$$d(\mu_t^0, \mu) \geq |\int f \, d\mu_t^0 - \int f \, d\mu| = H|g(\sigma_t^2) - g(\sigma_\infty^2)| \quad ;$$

g is derivable at point $\sigma^2 > 0$ with

$$g'(\sigma^2) = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}}\phi(x)e^{-\frac{x^2}{2\sigma^2}}x^{-\frac{3}{2}}(-\frac{1}{2} + (\sigma^2)^{-1}x^2) \, dx.$$

Since $-\frac{1}{2} + (\sigma^2)^{-1}x^2 \leq -\frac{1}{4}$ on the support of ϕ and since ϕ is positive in the neighbourhood of zero, it comes that $g'(\sigma^2) < 0$. Then

$$|g(\sigma_t^2) - g(\sigma_\infty^2)| \sim |g'(\sigma^2)||\sigma_t^2 - \sigma_\infty^2| = |g'(\sigma^2)|\|\Phi_t - \Phi_\infty\|_1,$$

which implies (21).

□

Remark 6. *In this remark, we would to explain that the set of suitable initial conditions u is relatively large.*

1. It is easy to construct nonzero sequences u satisfying to the assumption $\inf\{\hat{J}(z); z \in \text{spec } u\} > 0$: let $z_1, \ldots, z_p$ some points of $\mathbb{U}$ such that

$$\forall i \in [1, \ldots, p] \quad \hat{J}(z_i) > 0.$$

   Given complex numbers $\lambda_1, \ldots, \lambda_p$, the sequence u defined by

$$u_n = \text{Re } (\sum_{i=1}^{p} \lambda_i (z_i)^n) \tag{22}$$

   satisfies

$$\text{spec } u \subset \{z_1, \ldots, z_p\},$$

   which proved the desired assertion.

2. If K is a compact subset of $\mathbb{U}$ on which $\hat{J}$ does not vanish, every sequence defined like in (22), with z_i in K belongs to $E_{J,\tau}$, where $\tau = \inf_{z \in K} \hat{J}(z)$. If we prove that $E_{J,\tau}$ is closed for the $*$-weak topology of B_p, it follows that every weak limit of such sequences belongs to $E_{J,\tau}$ and then satisfies $\inf\{\hat{J}(z); z \in \text{spec } u\} > 0$.

   Lemma 18. *$E_{J,\tau}$ is a closed subspace for the $*$-weak topology of B_p.*

   Démonstration. Let $(u_n)_{n \geq 1}$ be a sequence in $E_{J,\tau}$ which converges to $u \in B_p$. For each $n \geq 1$, supp $u_n \subset F = \{z \in \mathbb{U}, \hat{J}(z) \geq \tau\}$. In other words, if O' is the open set $\mathbb{U} \backslash F$, we have for each $n \geq 1$, $O' \subset \cup_{O \in \mathcal{O}_{u_n}} O$. By lemma 14 $\cup_{O \in \mathcal{O}_{u_n}} O \in \mathcal{O}_{u_n}$, but it is easy to see that if O_1 and O_2 are open sets satisfying $O_1 \subset O_2$, then $O_2 \in \mathcal{O}_u \Longrightarrow O_1 \in \mathcal{O}_u$. Then, $O' \in \mathcal{O}_{u_n}$.
   Now let $f \in A_p$ such that supp $\hat{f} \subset O'$.
   For every n holds $\tilde{u_n}(f) = 0$, and then passing up to the limit, we get $\tilde{u}(f) = 0$. Hence $O' \in \mathcal{O}_u$, *i.e.* supp $u \subset F$, or equivalently $u \in E_{J,\tau}$. This proves that $E_{J,\tau}$ is closed for the $*$-weak topology of B_p. □

2.6 Extension to non-deterministic initial conditions

The previous results are formulated for deterministic initial conditions. However, it is not difficult to see that for any initial measure ν such that $\nu(\{x \in B_{p,0}; \Upsilon_J(x) > 0\}) = 1$, we have always $T_t \nu \to \mu_\infty$. If we want estimate the rate of convergence, we must control the speed with which $e^{-t\frac{J}{2}}\zeta$ vanishes, since $d(\mu_t^0, \mu_\infty)$ is controlled independently.

In case of uniqueness of the Gibbs measure and integrability of $\|\zeta\|_E$, inequalities

$$d(P_{e^{-t\frac{J}{2}}\zeta}, \delta_0) \leq \mathbb{E}\|e^{-t\frac{J}{2}}\zeta\|_{B_{p,0}} \leq \mathbb{E}\|\zeta\|_{B_{p,0}}\|e^{-t\frac{J}{2}}\|_{A_p}$$

together with lemma 11 with $\phi = e_0$, ensure an exponential rate.

The following result, including the phase transition case, is much finer.

Theorem 4. *Let J be a potential such that $\hat{J} \in C^{2k}(\mathbb{U}, \mathbb{C})$, where k is an integer satisfying to $2k > p + \frac{d}{2}$. We suppose that $\|\zeta\|_{B_{p,0}}$ is integrable and that there exist positive reals K and β such that*

$$P(\Upsilon_J(\zeta) < r) \leq K r^{\beta}.$$

Now, we have

$$d(P_{e^{-t\frac{J}{2}}\zeta}, \delta_0) = O((\frac{\ln t}{t})^{\beta}).$$

Démonstration. Let $f \in R_2$. For $x > 0$, one has

$$\begin{aligned}
|\mathbb{E}f(e^{-t\frac{J}{2}}\zeta) - f(0)| &= |\mathbb{E}(f(e^{-t\frac{J}{2}}\zeta) - f(0))\mathbb{1}_{\{\Upsilon_J(\zeta)<x\}} + \mathbb{E}(f(e^{-t\frac{J}{2}}\zeta) - f(0))\mathbb{1}_{\Upsilon_J(\zeta)\geq x)}| \\
&\leq 2P(\Upsilon_J(\zeta) < x) + \mathbb{E}\|e^{-t\frac{J}{2}}\zeta\|_{B_{p,0}}\mathbb{1}_{\{\Upsilon_J(\zeta)\geq x\}} \\
&\leq 2Kx^{\zeta} + K'_{J,k,p}(\frac{t}{2})^{4k}e^{-\frac{tx}{2}}\mathbb{E}\|\zeta\|_{B_{p,0}}
\end{aligned}$$

(The last inequality follows from the second part of lemma 16.)
We now choose $x = 2(4k+\beta)\frac{\ln t}{t}$: then

$$|\mathbb{E}f(e^{-t\frac{J}{2}}\zeta) - f(0)| \leq 2K(2(4k+\beta))^{\beta}(\frac{\ln t}{t})^{\beta} + \frac{K'_{J,k,p}\mathbb{E}\|\zeta\|_{B_{p,0}}}{2^{4k}}\frac{1}{t^{\beta}}.$$

Since this is true for each $f \in R_2$, the proof is done. □

3 Invariant measures are Gibbsian

The goal of this section is to determinate the set of invariant measures for the dynamics defined in (1), that is, using the notations introduced in (11), to determinate the measures $\nu \in \mathcal{P}(B_{p,0})$ such that for each $t \geq 0$, $\quad T_t\nu = \nu$.

Theorem 5. *Let p be positive. We suppose that J is a potential and k an integer such that $2k > \frac{d}{2} + p$ and $\hat{J} \in C^{2k}(\mathbb{U}, \mathbb{C})$.*

Then, for $\nu \in \mathcal{P}(B_{p,0})$, the following assertions are equivalent :

1. $\exists \mu \in \mathcal{P}(B_{p,0}) \quad T_t\mu \to \nu$
2. $\forall t \geq 0, \quad T_t\nu = \nu$
3. $\exists m \in \mathcal{P}(B_{p,0}) \quad m(\ker J) = 1$ *and* $\nu = m * \mu_\infty$, *where* μ_∞ *is the Gaussian centered measure with spectral density* $\frac{1}{\hat{J}}$.
4. ν *is a Gibbs measure with respect to the potential* J.

Démonstration. The implication 1. $\Rightarrow$ 2.is classic : for any fixed $t \geq 0$, the identity $T_{t+s}\mu = T_tT_s\mu$ holds. By (11), the process is Fellerian and, when s tends to infinity, we get $T_t\nu = \nu$. The converse implication 2. $\Rightarrow$ 1. is obvious.

Let us prove 3. $\Rightarrow$ 2. First, using 1. $\Rightarrow$ 2. with $\mu = \delta_0$, we see that μ_∞ is invariant under the dynamic. Now, if $\nu \in \mathcal{P}(B_{p,0})$ satisfies $\nu = m * \mu_\infty$ with $m(\ker J) = 1$, we obtain

$$\begin{aligned} T_t\nu &= \exp(-\frac{t}{2}J)(m * \mu_\infty) * \mu_t \\ &= \exp(-\frac{t}{2}J)m * \exp(-\frac{t}{2}J) * \mu_\infty * \mu_t \\ &= \exp(-\frac{t}{2}J)m * T_t\mu_\infty \\ &= m * \mu_\infty \\ &= \nu \end{aligned}$$

We now prove 2. $\Rightarrow$ 3. Let μ be another invariant measure. We will compare μ and μ_∞. It's easy to build a probability space $(\Omega, \mathcal{A}, P)$ and P-independent processes X_0 and W such that $P_{X_0} = \mu_\infty$ and W is as before an infinite Brownian motion under P. We define

$$X_t = e^{-t\frac{J}{2}}X_0 + W_t - \frac{J}{2}\int_0^t e^{-(t-s)\frac{J}{2}}W_s \, ds \tag{23}$$

Since μ is invariant, we have

$$\forall t \geq 0 \quad \mu = P_{X_t} = P_{e^{-\frac{t}{2}}X_0} * \mu_t^0 \tag{24}$$

We have $(X_t)_{t\geq 0}$ is tight because $P_{X_t} = \mu$. $(W_t - \frac{J}{2}\int_0^t e^{-(t-s)\frac{J}{2}}W_s \, ds)_{t\geq 0}$ is also tight because its law under P is μ_t^0 which converges to μ_∞. Since

$$e^{-t\frac{J}{2}}X_0 = X_t - (W_t - \frac{J}{2}\int_0^t e^{-(t-s)\frac{J}{2}}W_s \, ds), \tag{25}$$

it follows that $(e^{-t\frac{J}{2}}X_0)_{t\geq 0}$ is tight. Since μ is invariant, we have

$$\forall t \geq 0 \quad \mu = P_{X_t} = P_{e^{-\frac{t}{2}}X_0} * \mu_t^0$$

Every limit point m of $(P_{e^{-\frac{t}{2}}X_0})_{t\geq 0}$ is such that $\mu = m * \mu_\infty$ but using the characteristic functionals, it is easy to see that there this equation has at most one solution. Since $(e^{-t\frac{J}{2}}X_0)_{t\geq 0}$ is tight, it follows that it is convergent. Then, its limit m satisfies to

$$\forall t \geq 0 \quad e^{-\frac{t}{2}}m = m \tag{26}$$

and

$$\mu = m * \mu_\infty. \tag{27}$$

It remains to proof that the support of m is included in $\ker J$, or $Jm = \delta_0$: actually, we will prove that every measure on $\mathcal{P}(B_{p,0})$ with is invariant under the flow $(\exp(-tJ))_{t\geq 0}$ verifies $Jm = \delta_0$. Remark that such a measure is invariant under the flow $(\exp(tJ))_{t\geq 0}$ too.

In a first time, we will only prove that $J^{2k}m = \delta_0$.

Let $x \in \operatorname{supp} m$, $\varepsilon > 0$ and denote by $B(x,\varepsilon)$ the ball in $B_{p,0}$ with center x and radius ε. We have $\mu(B(x,\varepsilon)) > 0$. Since μ is invariant under $\exp J$, Poincaré's lemma ensures that there exists
$C \subset B(x,\varepsilon)$ with $\mu(C) = \mu(B(x,\varepsilon)) > 0$ and such that for each $y \in C$, the sequence $(\exp(nJ)y)_{n\geq 0}$ returns infinitely often in A. Since $\mu(C) > 0$, C is nonempty. Let $x_\varepsilon \in C$: we have

$$\underline{\lim}_{n\to\infty} \| \exp(nJ)x_\varepsilon - x\| \leq \varepsilon$$

and

$$\underline{\lim}_{n\to\infty} \| \exp(nJ)x_\varepsilon - x_\varepsilon\| \leq 2\varepsilon.$$

Lemma 19. *For $K \in A_p$ such that $\hat{K}(\mathbb{U}) \subset \mathbb{R}$, let us define*

$$h(K) = \sum_{j=1}^{+\infty} \frac{1}{j!} K^{j-1}. \tag{28}$$

Then, we have

$$\exp(K) = I + Kh(K),$$

and $h(K)$ is invertible in the Banach algebra A_p.

Démonstration. The first assertion is clear. We have

$$\widehat{h(K)} = \sum_{j=1}^{+\infty} \frac{1}{j!} \hat{K}^{j-1}. \tag{29}$$

Suppose that there exists $z \in \mathbb{U}$ such that $\widehat{h(K)}(z) = 0$ Since

$$\exp \hat{K} = 1 + \hat{K}\widehat{h(K)}, \tag{30}$$

we get $\hat{K}(z) \subset 2i\pi\mathbb{Z}$. But $\hat{K}$ is supposed to take only real values, so $\hat{K}(z) = 0$. But, by (29), it follows that $\widehat{h(K)}(z) = 1$ and we get a contradiction.

So, $\widehat{h(K)}$ does not vanish on $\mathbb{U}$ and, by lemma 2, $h(K)$ is invertible.

□

Since $k \geq 1$, we can write for each $t > 0$:

$$J^{2k} = \frac{1}{t} J^{2k-1} h(tJ)^{-1} (\exp(tJ) - I) \tag{31}$$

Then

$$\|J^{2k} x_\varepsilon\|_{B_p} \leq \|\frac{1}{t} J^{2k-1} h(tJ)^{-1}\|_{A_p} \; \|(\exp(tJ) - I) x_\varepsilon\|_{B_p} \tag{32}$$

So, if we assume that

$$M = \overline{\lim}_{t\to\infty} \|\frac{1}{t} J^{2k-1} h(tJ)^{-1}\|_{A_p} < +\infty, \tag{33}$$

we will have

$$\|J^{2k} x_\varepsilon\|_{B_p} \leq M \underline{\lim}_{t\to\infty} \|(\exp(tJ) - I) x_\varepsilon\|_{B_p} \leq 2M\varepsilon, \tag{34}$$

and then

$$\begin{aligned} \|J^{2k} x\| &\leq \|J^{2k}(x - x_\varepsilon)\|_{B_p} + \|J^{2k} x_\varepsilon\|_{B_p} && (35)\\ &\leq \|J^{2k}\|_{A_p} \|x - x_\varepsilon\|_{B_p} + 2M\varepsilon && (36)\\ &\leq (2M + \|J^{2k}\|_{A_p})\varepsilon, && (37) \end{aligned}$$

Since ε is arbitrary, $J^{2k} x = 0$ holds for each $x \in \operatorname{supp} \mu$, or, in other words, $\mu(\ker J^{2k}) = 1$.
Now, let $N \in \mathbb{N}$. We now apply Poincaré's lemma to the set $B(0, N) \cap \ker J^{2k}$: there exists $C_N \subset B(0, N) \cap \ker J^{2k}$ such that $\mu(C_N) = \mu(B(0, N) \cap \ker J^{2k})$ and such that for each $y \in C_N$, $(\exp(tJ)y)_{t\geq 0}$ returns infinitely often in C_N. But, since $C_N \subset \ker J^{2k}$, we have

$$\exp(tJ)y = \sum_{n=0}^{2k-1} t^n \frac{J^n y}{n!}. \tag{38}$$

If $n_0 = \max\{n \in \mathbb{N}; J^n y \neq 0\}$, we have when t tends to infinity :

$$\exp(tJ)y \sim t^{n_0} \frac{J^{n_0} n y}{n_0!} \tag{39}$$

Since $(\exp(tJ)y)_{t\geq 0}$ returns infinitely often in the bounded set C_N, we have necessary $n_0 = 0$. Then, $Jy = 0$ and $C_N \subset \ker J$. Now, if we define $\Omega' = \cup_{N\in\mathbb{N}} C_N$, we have $\Omega' \subset \ker J$ and

$$\mu(\Omega') \geq \sup_{N\in\mathbb{N}} \mu(C_N) = \sup_{N\in\mathbb{N}} \mu(B(0,N)\cap \ker J^{2k}) = \sup_{N\in\mathbb{N}} \mu(B(0,N)) = 1 \quad (40)$$

We now have to prove (33). We need some technical lemmas.

Lemma 20. *Define a function ϕ on $]0,+\infty[$ by*

$$\phi(x) = \frac{1}{e^x - 1}.$$

Then, for each $n \geq 0$, we have the following estimate at infinity :

$$\phi^{(n)}(x) = O(e^{-x}).$$

Démonstration. If we define $g(x) = \frac{x}{1-x}$, then $\phi(x) = g(e^{-x})$. Then, we can compute $\phi^{(n)}(x)$ using the formula of Faa di Bruno :

$$\phi^{(n)}(x) = \sum \frac{n!}{q_1!\dots q_n!} g^{(\sum_i q_i)}(e^{-x}) \frac{1}{1!\dots n!} e^{-(\sum_i q_i)x} \quad (41)$$

where the summation extends to the non-negative integers such that $\sum_{i=1}^n iq_i = n$. Since g has bounded derivatives in the neighbourhood of 0, the lemma follows. □

Lemma 21. *Let $j \geq 1$ and $\Psi_j : \mathbb{R} \to \mathbb{R}$ defined by $\Psi_j(x) = x^j\phi(x)$. We have $\Psi_j \in C^\infty(\mathbb{R},\mathbb{R})$ and*

$$\forall n \in \mathbb{N} \quad \sup_{x\in\mathbb{R}^+} |\Psi_j^{(n)}(x)| < +\infty.$$

Démonstration. It's an easy consequence of lemma 20 with help of Leibnitz's formula. □

Now, we can write

$$\frac{1}{t}\widehat{J^{2k-1}h(tJ)^{-1}} = \frac{1}{t^{2k}}\Psi_{2k}(t\hat{J})$$

Then, by lemma 10, there exists $K > 0$ such that

$$\|\frac{1}{t}J^{2k-1}h(tJ)^{-1}\|_{A_p} \leq \frac{K}{t^{2k}}\|(1-\Delta)^k\Psi_{2k}(t\hat{J})\|_{L^2(\mathbb{U})}$$

But

$$\begin{aligned}
\|(1-\Delta)^k \Psi_{2k}(t\hat{J})\|_{L^2(\mathbb{U})} &\leq \|(1-\Delta)^k \Psi_{2k}(t\hat{J})\|_\infty \\
&\leq K' \sup_{0\leq i\leq 2k; x\in\mathbb{R}} |\Psi_{2k}^{(i)}(x)| \; \|t\hat{J}\|_{D^{2k}} \\
&\leq K' \sup_{0\leq i\leq 2k; x\in\mathbb{R}} |\Psi_{2k}^{(i)}(x)| \; \|\hat{J}\|_{D^{2k}}(1+t^{2k})
\end{aligned}$$

So (33) is proved. Since the equivalence between 3. and 4. follows from (9), the proof of Theorem 5 is now complete. □

Theorem 6. *Let p be positive. We suppose that $J \in A_p$ is a potential and k an integer such that $2k > \frac{d}{2} + p$ and $\hat{J} \in C^{2k}(\mathbb{U}, \mathbb{C})$. Moreover, we suppose that $\hat{J} > 0$ on $\mathbb{U}$. Then,*

$$\{\nu \in \mathcal{P}(B_{p,0}); \forall t \geq 0 \quad T_t\nu = \nu\} = \mathfrak{G}_J \cap \mathcal{P}(B_{p,0}) = \{\mu_\infty\}.$$

Démonstration. The first equality has been proved in Theorem 5. Obviously, $\mathfrak{G}_J \cap \mathcal{P}(B_{p,0}) \subset \mathfrak{G}_J \cap \mathcal{P}(B_p)$. But by Proposition 2, $\mathfrak{G}_J \cap \mathcal{P}(B_p) = \{\mu_\infty\}$. Since $\mu_\infty \in \mathfrak{G}_J \cap \mathcal{P}(B_{p,0})$, the result follows. □

Example : Let $m \geq 0$ and define J by

$$J(i) = \begin{cases} 1+m & \text{if } i = 0 \\ -\frac{1}{2d} & \text{if } |i| = 1 \\ 0 & \text{else} \end{cases}$$

Since $\hat{J} \in C^\infty(\mathbb{U}, \mathbb{R})$, the regularity assumptions are satisfied.
For $m > 0$, J is the so-called *harmonic model with mass*. By a direct computation, we see that $\hat{J} > 0$ on $\mathbb{U}$. Then, for each $p > 0$, the assumptions of Theorem 6 are fulfilled and the unique Gibbs measure with support in $B_{p,0}$ is the unique invariant measure for the dynamic with support in $B_{p,0}$.
For $m = 0$, J is the so-called *massless harmonic model*. By proposition 1, $\mathfrak{G}_J$ is non-empty if and only if $\frac{1}{\hat{J}}$ is integrable, that is if and only if $d \geq 3$. Then, the invariant measures for the dynamic whose support is in $B_{p,0}$ are the associated Gibbs measures with support in $B_{p,0}$. Precisely, these are the measures obtained by convolution of μ_∞ by every measure whose support is included in the set of harmonic sequences which are in $B_{p,0}$.

Remark 7. *It is interesting to compare Theorem 5 with the finite dimensional case, which has been studied by Zakai and Snyders [25] : they proved that the stationary measures associated to d-dimensionnal s.d.e.*

$$X_t = X_0 + \int_0^t AX_s \, ds + BW_t.$$

are obtained as convolution of a certain measure μ and an arbitrary invariant measure for the deterministic dynamic

$$x_t = x_0 + \int_0^t Ax_s \, ds.$$

This is the way used in the present paper. Of course, an invariant measure for the deterministic dynamic has not necessary all its support in $\ker A$ *; see for example* $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ *and* $B = I_2$ *: in this case, one can take* $\mu = \mathcal{N}(0, I_2)$ *and each radial measure on* $\mathbb{R}^2$ *is invariant under the semi-group because* $\exp(-tA)$ *is composed by rotations. Such a phenomenon of periodicity can not happen here, because the spectrum of* J*, which is a symmetric operator, is real (it is equal to* $\hat{J}(\mathbb{U})$*, cf [12]). Note that the given example for* A *is antisymmetric – and therefore has purely imaginary spectrum. Since it is not symmetric, it can not be obtained as the gradient of an interaction potential.*

Remark 8. *Sometimes, a moment condition on the invariant measure is preferable to a support condition. It can be done easily using the lemma of Borel-Cantelli. For example, if* $J \in \mathcal{S}(\mathbb{Z}^d)$ *and if one wants to show that a stationary measure* μ *is Gibbs, it suffices to show the existence of an* $\alpha > 0$ *such that*

$$\sup_{i \in \mathbb{Z}^d} \mathbb{E}_\mu |X_i|^\alpha < +\infty.$$

4 Conclusion

In this paper, we first represented the extremal Gibbs measures associated to a quadratic potential as temporal limits of an infinite linear system of stochastic differential equations with some deterministic initial conditions.

- Uniqueness of such a Gibbs measure with support in $B_{p,0}$ then coincides with the ergodicity of the associated gradient dynamics, in other words, the absence of phase transition corresponds to the case where the system converges to a limit which is the same for all initial conditions.
- This way to get Gibbs measures may be seen as complementary to the classical D.L.R. approach which consists in getting extremal Gibbs measures as spatial limits of a collection of finite-dimensional local specifications with a fixed sequence of external conditions. Kondratiev and Sokol have already pointed out this fact in [20].
- In case of phase transition, we do exhibit for each pure phase an affine space of initial conditions as domain of attraction. Indeed, each pure

phase can be written $\tau_s\mu_\infty$, with $s \in \ker J$, and is limit of the diffusion starting from $u+s$, with $\Upsilon_J(u) > 0$.

Moreover, we proved that presence or absence of phase transition has an immediate consequence on the rate of convergence for the dynamics : it is exponential in case of uniqueness and polynomial in case of phase transition ; we show that the order d_0 of an unique root of $\hat{J}$ directly determines the rate of convergence, which is

$$\frac{1}{t^{\frac{d}{d_0}-1}}.$$

In the last section, we proved that each invariant measure is a Gibbs state – and therefore, that every limit of the dynamics too. It is, for the Gaussian case, an answer for the still open conjecture about existence of invariant non Gibbsian measures for non-linear infinite-dimensional gradient systems.

Références

[1] R.L. Dobrushin. Gaussian random fields - gibbsian point of view. In R.L. Dobrushin and Ya.G. Sinai, editors, *Multicomponent random systems*, volume 6 of *Advances in Probability and Related Topics*. Dekker, 1980.

[2] A. Araujo and E. Giné. *The Central Limit Theorem for Real and Banach Valued Random Variables*. Wiley, 1980.

[3] H. Cartan. *Cours de calcul différentiel*. Hermann, 1977.

[4] P. Cattiaux, S. Rœlly, and H. Zessin. Une approche gibssienne des diffusions browniennes infini-dimensionnelles. *Probab. Th. Rel. Fields*, 104 :147–179, 1996.

[5] G. Da Prato and J. Zabczyk. *Stochastic Equations in Infinite Dimensions*. Cambridge, 1992.

[6] G. Da Prato and J. Zabczyk. Convergence to equilibrium for classical and quantum spin systems. *Probab. Th. Rel. Fields*, 103 :529–552, 1995.

[7] J.D. Deuschel. Invariance principle and empirical mean large deviations of the critical Ornstein-Uhlenbeck process. *The Annals of Probability*, 17 :74–90, 1989.

[8] H. Doss and G. Royer. Processus de diffusion associé aux mesures de Gibbs sur $\mathbb{R}^{\mathbb{Z}^d}$. *Z. Wahrsch. Verw. Gebiete*, 46 :107–124, 1978.

[9] J. Fritz. Infinite lattice systems of interacting diffusion processes, existence and regularity properties. *Z. Wahrsch. Verw. Gebiete*, 59 :291–309, 1982.

[10] J. FRITZ. Stationary measures of stochastic gradient systems, infinite lattice models. *Z. Wahrsch. Verw. Gebiete*, 59 :479–490, 1982.

[11] O. GARET. *Mesures de Gibbs gaussiennes et dynamiques aléatoires associées sur* $\mathbb{R}^{\mathbb{Z}^d}$. PhD thesis, Université de Lille I, 1998.

[12] O. GARET. Gibbsian fields associated to exponentially decreasing quadratic potentials. *Ann. Inst. H. Poincaré, Probabilités*, 35 :387–415, 1999.

[13] H.-O. GEORGII. *Gibbs Measures and Phase Transitions.* Walter de Gruyter, 1988.

[14] P. HALMOS. *A Hilbert Space Problem Book.* Springer-Verlag, 1982.

[15] R. HOLLEY and D. STROOCK. Diffusions on an infinite dimensional torus. *J. Funct. Anal.*, 42 :29–63, 1981.

[16] J.P KAHANE. *Séries de Fourier absolument convergentes.* Springer, 1970.

[17] Y. KATZNELSON. *An introduction to harmonic analysis.* Dover, 1976.

[18] H.-R. KÜNSCH. Gaussian Markov random fields. *J. Fac. of Science of Univ. of Tokyo*, 26 :53–73, 1979.

[19] H.-R. KÜNSCH. *Reellwertige Zufallsfelder auf einem Gitter : Interpolationsprobleme, Variationsprinzip und statistische Analyse.* PhD thesis, Zürich, 1980.

[20] Yu. G. KONDRATIEV and T.A. SOKOL. Gaussian measures and stochastic quantification. *Selecta Mathematica formerly Sovietica*, 13(3) :225–238, 1994.

[21] G.K. PEDERSEN. *Analysis Now.* Springer-Verlag, 1995.

[22] Yu. A. ROSANOV. On Gaussian fields with given conditional distributions. *Th. Prob. Appl.*, 12 :433–443, 1967.

[23] G. ROYER. Processus de diffusion associé à certains modèles d'Ising à spins continus. *Z. Wahrsch. Verw. Gebiete*, 46 :165–176, 1979.

[24] T. SHIGA and A. SHIMIZU. Infinite dimensional stochastic differential equations and their applications. *J. Math. Kyoto. Univ.*, 20-3 :395–416, 1980.

[25] M. ZAKAI and J. SNYDERS. Stationary probability measures for linear differential equations driven by white noise. *J. Diff. Equ.*, 8 :27–33, 1970.